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Mozart's Use of the Golden Ratio: His Mathematical Background Exposed in his Piano Sonatas

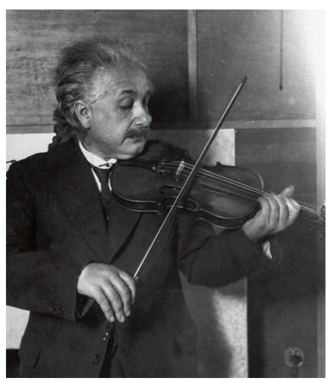
After centuries of investigation, scholars have found a strong correlation between music and mathematics is evident. It is widely known that the Greeks were fascinated with mathematical concepts and their undeniable relationship with music. Since the time period of the ancient Greeks, music has been perceived as a mathematical art. It was believed by the Greeks that there is a divine quality in numbers, some more perfect than others. Hence, it was believed that the music of the universe was driven by numbers. Going further, Galileo, an astronomer, believed that the language of mathematics made up the entirety of the universe. In addition, a treatise on string theory was written by the lutenist Vincenzo, Galileo's father.[1] An author, Rajen Barua, offers how miraculous the magnitude to which society and science are driven by mathematical concepts.[2]

In addition to theories of the cosmos, music is based on mathematical relationships in logical concepts such as scales, chords, octaves, and keys. Further, the theorist Pythagoras believed that the simpler relative frequencies of musical notes were more pleasing to the ear than those that were more complex. Basically, he started the belief that in order for music to be aesthetically pleasing, the notes of the musical scale must be based on the perfect fifth ratio.[3] Based upon his belief, Pythagoras created the tuning of the musical scale based upon the ratio frequencies of whole number intervals in mathematical harmonics.[4] Furthering the relationship between music and mathematics, it has been argued that the roots of mathematics and music are closely connected. Given its properties of harmony and order, mathematics has a pleasing structure. Though there is no sound transmission, it can be argued that

[1] Patrick Hunt, "Mozart and Mathematics," *Electrum Magazine,* 2013.
[2] Rajen Barua, "Music, Mathematics and Mozart," *Gonitsora*, 2011.
[3] Barua, "Music, Mathematics and Mozart," 2011.
[4] Hunt, "Mozart and Mathematics," 2013.

these properties make mathematics musical. Based on its quality, rhythm, melody, frequency, amplitude, style, and form, it can be argued that music is mathematical, though it does not consist of mathematical notation.[5]

Additionally, many mathematicians also thrive as musicians and many musicians study mathematics. For example, a founder of the Juilliard Quartet, Virtuoso violist Raphael Hillyer, received a degree in mathematics.[6] In addition, Max Born, a quantum mechanics pioneer, mathematical physicist, and statistical interpreter of the quantum mechanics wave function, played Bach's piano works on a daily basis.[7] Leonhard Euler, an incredible mathematician and theorizer of musical consonance, had a strong passion for music and regularly invited composers to perform at his home.[8] Finally, Albert Einstein, famously known for his mathematical and scientific genius, played the violin.[9]



Einstein playing violin[10]

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[5] Barua, "Music, Mathematics and Mozart," 2011.
[6] Hunt, "Mozart and Mathematics," 2013.
[7] Hunt, "Mozart and Mathematics," 2013.
[8] Gerard Assayag, "Mathematics and Music: A Diderot Mathematical Forum," *Musical Patterns,* 2002.
[9] Barua, "Music, Mathematics and Mozart," 2011.
[10] Ray Moore, "Physicist Albert Einstein, seen here playing the violin,"12/8/2014,website, http://ualrpublicradio.org/post/albert-einstein-physicist-and-violinist

In general, the composing of classical music can be compared to applied mathematics. In the works of a genius, such as Mozart or Bach, harmonics, ordered melody, and other chord progression elements have the possibility of approaching emotional equations, which would explain why physicists and mathematicians enjoy the music of Bach and Mozart above many other composers.[11]

The best works of a mathematician or musician seem to generally be created when they are young, bringing Mozart into the equation. Barua offers the opinion that mathematicians and performing musicians tend to mature young and exhibit child genius in these disciplines more than others. As a child prodigy, Mozart perfectly fits this description.[12] By the time he was six years old, he began composing pieces for the clavier and became completely absorbed in music instead of showing interest in childish activities.[13]

More specifically, the Golden Ratio further relates music to mathematics. The golden section, a precise division of two parts, was exposed at least as far back as 300 B.C. when it was described by Euclid the *Elements*, his major work.[14] However, there is some evidence that the golden section was thought of around 500 B.C. by Pythagoreans as well. Regardless of its discovery by humans, the oldest examples of this division appear in the proportions of nature[15] and are often thought as the most divine and aesthetically pleasing proportions.[16] Though this is opinionated, it can be stated that the effect of identical ratios has a fundamental way of unifying the structure.[17]

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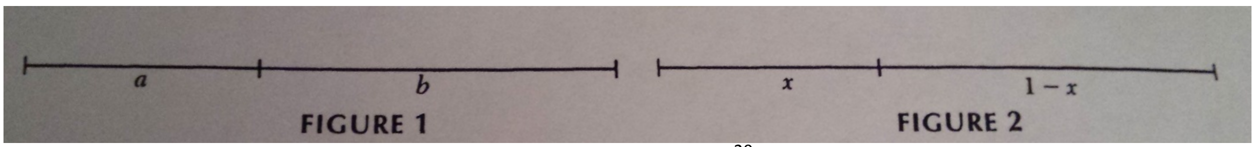
[11] Hunt, "Mozart and Mathematics*,"* 2013.
[12] Barua, "Music, Mathematics and Mozart," 2011.
[13] Franz Niemetschek, *Life of Mozart*, London: Leonard Hyman, 1798, 14.
[14] Mike May, "Did Mozart Use The Golden Section?" *American Scientist* 84 (1996), 118.
[15] May, "Did Mozart Use The Golden Section?," 118.
[16] John F. Putz, "The Golden Section and the Piano Sonatas of Mozart," *Mathematics Magazine* 68 (1995), 275.
[17] Putz, "The Golden Section and the Piano Sonatas of Mozart," 275.

The division of the Golden Ratio is *a/b = b/(a+b)*, where *a* and *b* are two unequal line segments such that the length of the shorter segment *a* is to the length of the longer segment *b* just as the length of the longer segment is to the whole.[18] In other words, imagine a line with the length of one unit and divide that line into two unequal segments. Label the shorter segment as x and the longer as (1-x); therefore, the ratio of the shorter to the longer segment is equivalent to the ratio of the longer segment to the line as a whole. Thinking in those terms, the ratio now appears conveniently as x/(1-x) = (1-x)/1.[19]



The Golden Section[20]

This equality leads to a quadratic equation, and after solving for *x* and substituting that value into the equality for *x*, a numerical value for the ratio of about 0.618 is created.[21]

Generally speaking, the natural quality of the Golden Ratio has influenced many composers, artists, and architects. In many ways, art is the imitation of nature, so this is to be expected. In the *Mathematics Magazine*, John F. Putz states that "ubiquitous in nature, the golden section embodies its elegant proportion in the starfish and the chambered nautilus, in the pine cone and the sunflower, and in leaf patterns along the stems of plants"[22], further expressing the relationship of the Golden Ratio with nature. Contemplated by musicologists for decades, this proportion is evident in the piano sonatas of Mozart. Some theorists and researchers think this occurs by coincidence, but I argue that there exists proof of Mozart's knowledge and deliberate use of the Golden Ratio in his piano sonatas.

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[18] Putz, "The Golden Section and the Piano Sonatas of Mozart," 275.
[19] May, "Did Mozart Use The Golden Section?" (1996), 118.
[20] Putz, "The Golden Section and the Piano Sonatas of Mozart," 275.
[21] May, "Did Mozart Use The Golden Section?" (1996), 118.
[22] Putz, "The Golden Section and the Piano Sonatas of Mozart," 275.

Before going in depth with Mozart's use of the Golden Ratio, one needs to understand the depth of Mozart's background with, and love for, mathematics. It has been proven by a multitude of biographers that Mozart had a strong passion for mathematics, especially Numerology.[23] Though he is known more famously for his musical compositions, that was not his only talent. In general, Mozart learned things very easily and enthusiastically as a result of his sensitive nature, including both composition and mathematics. Mozart's sister recalls that when Mozart was learning arithmetic, arithmetical problems was all he spoke or thought of and would even cover tables, floors, chairs, and walls with numbers.[24] She also recalls that when he had finished covering everything possible in his own home, he would then cover the neighbor's houses with figures as well.[25] Finally, she claims Mozart also asked her to send him arithmetical exercises and tables in a letter he wrote to her at the age of 14 while he was traveling as a composer.[26]

Mathematics was an integral portion of Mozart's brain, and figures were constantly on his mind even while composing. Throughout some of his compositions, there is evidence of arithmetical problems and mathematical equations in the margins on his manuscripts.[27] For example, in the *American Scientist*, Mike May states that Mozart jotted mathematical equations to help calculate his odds of winning the lottery in his manuscript of *Fantasia and Fugue in C Major*.[28] In addition, there exist pages of musical sketches where Mozart attempted to "figure out the sum which the chess player would have received from the King in the famous Oriental story".[29] These instances do not offer equations that have related

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[23] Barua, "Music, Mathematics and Mozart."
[24] Niemetschek, *Life of Mozart* (1798), 20.
[25] Putz, "The Golden Section and the Piano Sonatas of Mozart," 276.
[26] Putz, "The Golden Section and the Piano Sonatas of Mozart," 276.
[27] Barua, "Music, Mathematics and Mozart."
[28] May, "Did Mozart Use The Golden Section?" 118.
[29] Putz, "The Golden Section and the Piano Sonatas of Mozart," 276.

directly to his music, they show that Mozart was constantly concentrating on mathematics even when composing.

Not only was Mozart fascinated with mathematics, Mathematicians were also fascinated with Mozart. Since there was such a strong connection between mathematics and music in Mozart's brain, Patrick Hunt states that Mozart did not just write mathematical music by chance, but did so consciously.[30] As a result, Mozart's musical structures continue to fascinate many mathematicians. For example, mathematicians continue to speculate the use of the Fibonacci sequence in Mozart's piano sonatas, particularly in *his Piano Sonata #1 in C major* k279.[31] This is very probable since his sister Nannerl has stated that Mozart scribbled mathematical equations in the margins of his compositions, many of which mathematicians believe to be part of the Fibonacci sequence.[32]

Other mathematicians have also contemplated the use of musical symbolic number combinations in works of both Mozart and Bach. For example, Patrick Hunt believes there is musical gematria in Mozart's *Don Giovanni*:[33] "Leporello's 'catalogue aria' first recites the Don's conquests as adding up to an unstated 1,062 (640 in Italy + 231 in Germany + 100 in France + 91 in Turkey) then adds to this sum, 1,003 conquests in Spain, making a total of 2,065", exposing Mozart's conscious effort with the use of musical symbolic number combinations in the opera.[34] In addition, in Mozart's *Marriage of Figaro*, Patrick Hunt has realized that Figaro counts the measure of his imaged quarters to be shared with Susanna in his footsteps: "5, 10, 20, 30, 36, 43, the sum of which is 144 or 12 squared, as others like de Sautoy have pointed out, again noting it may not have any additional meaning, although coincidence

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[30] Hunt, "Mozart and Mathematics."
[31] Ibid.
[32] Ibid.
[33] Ibid.
[34] Ibid.

seems unlike Mozart", further proving Mozart's use of symbolic number combinations in his compositions.[35]

In continuation, mathematicians have also examined the mathematical symmetry of Mozart's music. For example, Mario Livio, an author and mathematical astrophysicist, examines Mozart's *Musical Dice Game Minuet* consisting of 16 measures "with the choice of one of eleven possible variations in measure endings from random selection, each possibility selected by a roll of 2 dice, with literally trillions of possible mirror combinations," expressing Mozart's love for symmetry.[36]

Further, mathematicians have concluded that many of the variations of his musical themes are like number games. For example, in his *Symphony #40 in G minor*, Hunt suggests "the developments and inversions of his musical themes are like contrapuntal and antiphonal number games between flute and violins, especially in bridging passages between measures 119 and following, again in fugal passages beginning in measures 150 ff & 160-220 in the first movement", further proving Mozart's conscious use of mathematics in his compositions.[37]

In terms of mathematical equations, mathematicians wonder if Mozart actually composed his pieces with mathematical equations, causing mathematics to play a very active role in Mozart's compositional success. Some researchers, such as Author Mario Livio, greatly support this inference. Livio studies the relationship between art and mathematics, and believes that both symmetry and elements of surprise are what attracts the human brain to art; in regard to Mozart, symmetry and

[35] Hunt, "Mozart and Mathematics."
[36] Ibid.
[37] Ibid.

harmonic surprises are the elements in which make up his compositions. Though musical, harmony and symmetry is also the foundation of mathematics.[38]

In general, Barua discusses that “this structural analysis of the music and its effects on the listener includes examples of the application of mathematics to music, the measurement of the effects of Mozart’s music, the application of the Golden ratio to Mozart’s musical structure, and an analysis of the application of mathematics to the musical structure of Mozart’s concerts”[39]. Though there is a great amount of number symbolism in the works of Bach, Beethoven, and others of that era, Mozart’s music seems to have the greatest effect in stimulating the mathematical portion of the brain, known as the Mozart effect.[40] Alfred Einstein, a famous scientist and mathematician, is particularly fascinated in Mozart’s music as a result of these effects. Einstein once stated, “while Beethoven created his music, Mozart’s was so pure that it seemed to have been ever-present in the universe, waiting to be discovered by the master”[41], further developing the idea that Mozart’s music was much more mathematical than the music of other composers of that era. In a biography Einstein wrote on Mozart, he stated that Mozart’s passion for mathematics continued to grow until the day he died, and he even decided to compose minuets mechanically with two-measure melodic fragments.[42]

Going in the direction of the Golden Ratio, even someone who has never heard any works by Mozart will find it sounding familiar because of Mozart’s memorable melodies and proportions.[43] Mozart is famously known for his balance and form, and his music is famously known for its beautifully symmetric proportions.[44] In regards to Mozart’s music, Henri Amiel stated that “the balance of the

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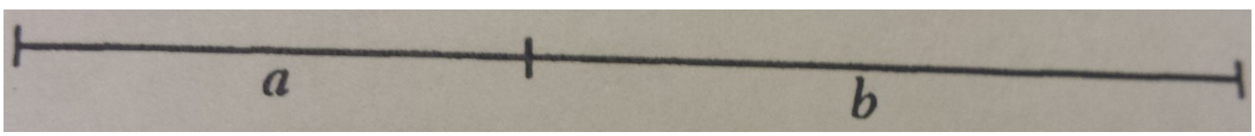
[38] Barua, “Music, Mathematics and Mozart,” 2011.
[39] Barua, “Music, Mathematics and Mozart,” 2011.
[40] Barua, “Music, Mathematics and Mozart,” 2011.
[41] Barua, “Music, Mathematics and Mozart,” 2011.
[42] Putz, “The Golden Section and the Piano Sonatas of Mozart,” 276.
[43] Putz, “The Golden Section and the Piano Sonatas of Mozart,” 276.
[44] Putz, “The Golden Section and the Piano Sonatas of Mozart,” 276.

whole is perfect" and Hanns Dennerlein thought of Mozart's music as "reflecting the most exalted proportions" and Mozart as having "an inborn sense for proportions".[45] In addition, Eric Blom also expressed his opinion that Mozart had "an infallible taste for saying exactly the right thing at the right time and at the right length".[46] How did Mozart's proportions manage to stand out so significantly compared to the music of other composers of his time? The answer is simple, Mozart used the Golden Ratio, said to create the most elegant proportions, when composing his works. Many of Mozart's works exhibit this proportion, but for this purpose, my main focus will be on his piano sonatas. The sonata-form used by Mozart consisted of two part: the Exposition and the Development and Recapitulation. In the Exposition, the musical theme is introduced, and in the Development and Recapitulation, that theme is developed, inverted, and varied throughout. It is my belief that these two sections were divided by the golden ratio with Development and Recapitulation as the longer segment.



Mozart's Sonata Form[47]

In this caption, let *a*=Mozart's Exposition and *b*=Mozart's Development and Recapitulation.

Here is a collection of Mozart's sonata movements using the Kochel cataloging system. If codas were present, they were not included as part of the second section.

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[45] Ibid, 276.
[46] Ibid, 276.
[47] Ibid, 275.

Köchel	a	b	$a+b$
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102
283, I	53	67	120
283, II	14	23	37
283, III	102	171	273
284, I	51	76	127
309, I	58	97	155
311, I	39	73	112
310, I	49	84	133
330, I	58	92	150
330, III	68	103	171
332, I	93	136	229
332, III	90	155	245
333, I	63	102	165
333, II	31	50	81
457, I	74	93	167
533, I	102	137	239
533, II	46	76	122
545, I	28	45	73
547a, I	78	118	196
570, I	79	130	209

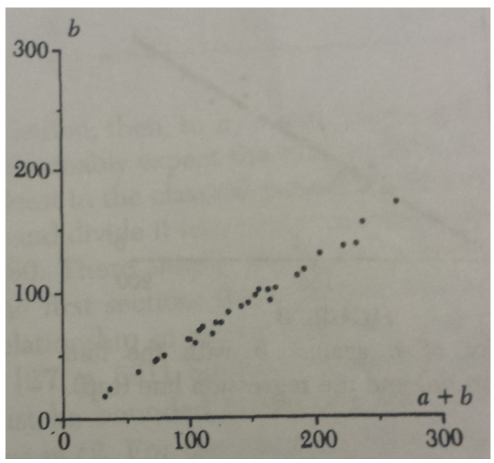
Table of Mozart's Sonatas Analyzed by their Proportions[48]

We see that the first movement of the first Sonata, 279, I, is 100 measures in length with a Development and Recapitulation section of length 62. Rounded to the nearest natural number, 100 multiplied by the Golden Ratio (.618) is equivalent to 62. Therefore, this a perfect golden section division; a movement consisting of 100 measures could not, in natural numbers, be divided any closer to the golden section than its division of 38 and 62.[49] The second movement of this piece also follows, a movement consisting of 74 measures could not be divided any closer to the golden section than its division of 28 and 46. However, there is some speculation as to whether or not the third movement was divided in the golden

[48] Putz, "The Golden Section and the Piano Sonatas of Mozart," 277.
[49] Putz, "The Golden Section and the Piano Sonatas of Mozart," 277.

section; the second section would have to be 98 for a perfect division instead of 102. Although, that division is still extremely close to that of the golden section.[50]

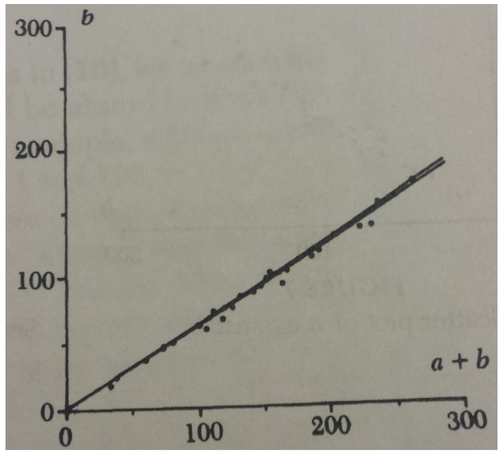
Obviously, analyzing these movements separately does not provide enough insight as to Mozart's use of the Golden Ratio. However, to use a visual aid in evaluating the consistency of these proportions, a scatter plot of *b* against *a+b* is provided. If Mozart divided the movements in relation to the golden section, then the data points will lie near the line y = .618x. Also, there should be a linearity to the data if Mozart was consistent.



Scatter plot of *b* against *a+b*[51]

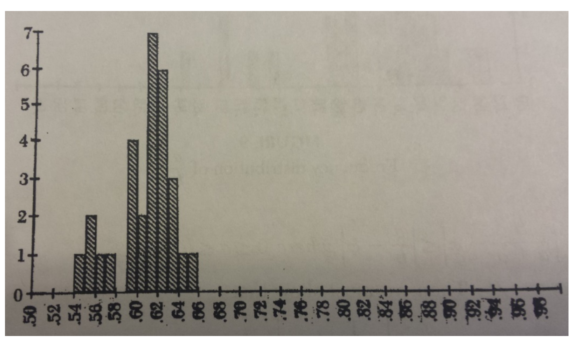
With an r^2 value, a value showing the percentage of total variation on the vertical axis explained by the horizontal axis, of 0.990, the degree of linearity of this data is incredibly high. Therefore, it is evident that Mozart was consistent. Further examining this data in terms of its relation to the golden section, the line y=0.618x and the regression line y=-0.003241+0.6091x are added to this scatterplot as shown below.

[50] Ibid, 277.
[51] Ibid, 278.



Scatter plot of *b* and *a* with the line y=0.618 (top) and the regression line (bottom)[52]

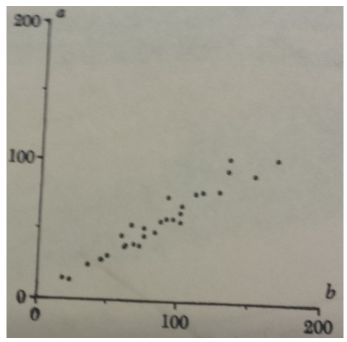
Since the measures in the sonata consist of natural numbers, the line y=0.618 is expectedly a bit above the regression line because of its slope; however, this line is barely differentiable from the regression line (the line of best fit), which has an alarmingly similar slope to begin with.[53] However, this still does not provide all of the necessary information needed to prove Mozart did indeed use the Golden Ratio in these sonatas. There still must be some way of calculating the centrality of 0.618 in this data. Therefore, a histogram of the ratio of b/(a+b) is provided below.



Frequency distribution of b/(a+b)[54]

[52] Putz, "The Golden Section and the Piano Sonatas of Mozart," 278.
[53] Putz, "The Golden Section and the Piano Sonatas of Mozart," 277.
[54] Putz, "The Golden Section and the Piano Sonatas of Mozart," 278.

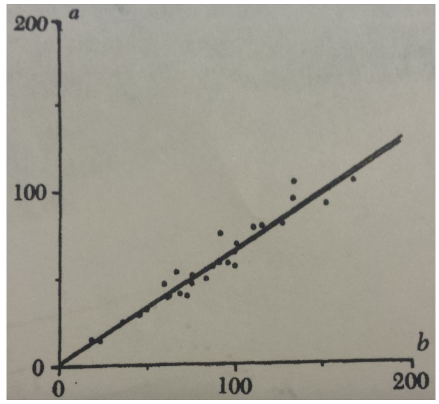
At this point it is not alarming, yet still impressive, that the data's centrality is clear in regards to the value of the ratio. John F. Putz states that this alone should be "impressive evidence that Mozart did, with considerable consistency, partition sonata movements near the golden section."[55] However, the data must be analyzed in yet another way to be thoroughly convincing. If these movements were truly divided by the Golden Ratio, then both a/b and b/(a+b) should be close to 0.618, not just b/(a+b). Therefore, provided below is a scatterplot showing the relationship between *a* and *b*.



Scatter plot of *a* against *b*[56]

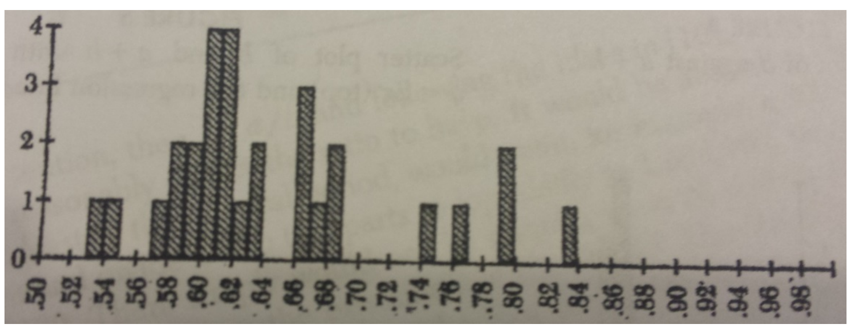
Though this data still looks relatively linear, it is not as linear as the relationship between *b* and *a+b*. Again, the data can be further examined in terms of its relation to the golden section with the line y=0.618x and the regression line which is now y=1.360+0.6260x. The scatterplot with the addition of these two lines is provided below.

[55] Putz, "The Golden Section and the Piano Sonatas of Mozart," 278
[56] Putz, "The Golden Section and the Piano Sonatas of Mozart," 279.



Scatter plot of *a* against *b* with the line y=0.618x (bottom) and the regression line (top)[57]

Though these two lines are not very differentiable, the r^2 of 0.938 does not express as strong of a correlation as the first. Although, an r^2 value of 0.938 is still extremely strong. However, a histogram is provided below to analyze the centrality of the data when comparing *a* to *b*.



Frequency distribution of a/b[58]

This histogram shows much more variance in the data than that of the one comparing *a* and *a+b*, suggesting less evidence for the centrality of the ratio. Though it is possible to selectively interpret any set of data toward the way you wish for it to appear, this is more mathematical than that. In actuality, John F. Putz discuses a theorem which proves that "what we have observed in these data is true for all

[57] Putz, "The Golden Section and the Piano Sonatas of Mozart," 279.
[58] Putz, "The Golden Section and the Piano Sonatas of Mozart," 279.

data; b/(a+b) is always nearer to [0.618] than is a/b"[59]; therefore, investigations must be confined to the ratio a/b. Provided below is the theorem discussed by Putz.

THEOREM. $\left|\frac{b}{a+b} - \varphi\right| \leq \left|\frac{a}{b} - \varphi\right|$ *where* $0 \leq a \leq b$.

Proof. Let $x = a/b$. Then we must show that

$$\left|\frac{1}{x+1} - \varphi\right| \leq |x - \varphi|$$

for all $x \in [0,1]$. Let $f(x) = 1/(x+1)$. By the Mean Value Theorem, for all $x \in [0,1]$ there is a $\xi \in (0,1)$ such that

$$|f(x) - f(\varphi)| = |f'(\xi)|\,|x - \varphi|.$$

Now $f'(x) = -1/(x+1)^2$ satisfies

$$\tfrac{1}{4} < |f'(x)| < 1$$

for $x \in (0,1)$. A simple calculation will show that φ is a fixed point of f, that is, that $f(\varphi) = \varphi$. So, for all $x \in [0,1]$,

$$\left|\frac{1}{x+1} - \varphi\right| \geq |x - \varphi|$$

with equality when $x = \varphi$.

We note, in passing, that the fixed-point algorithm of numerical analysis works on this principle, and that this theorem says that the ratio of consecutive terms of any Fibonacci-like sequence ($f_1 = a, f_2 = b, f_{n+2} = f_n + f_{n+1}$ with a and b not both zero) converges to φ.

Theorem[60]

Therefore, for any given pair *a* and *b*, where *a* is greater than or equal to zero and *b* is greater than or equal to *a*, the ratio b/(a+b) will always be closer to 0.618 then a/b will.[61] As a result of this proof, we must rely on the ratio of *a/b*. However, this ratio will show more variance by nature, so it must be determined what values should be expected of the ratio in order for the data to be significant. It is obvious that a composer of Mozart's time would not write a 200–measure long movement with a 10 measure exposition and a 190 measure Development and Recapitulation, but Quantz offers his opinion that in order for a pleasantly balanced proportion to occur, the exposition should be shorter than the development and recapitulation.[62] In mathematical terms, "if we let the length of the movement *m=a+b*

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[59] Putz, "The Golden Section and the Piano Sonatas of Mozart," 278,
[60] Putz, "The Golden Section and the Piano Sonatas of Mozart," 279.
[61] Putz, "The Golden Section and the Piano Sonatas of Mozart," 278.
[62] Putz, "The Golden Section and the Piano Sonatas of Mozart," 280.

be fixed, then *a* must be bounded below at some practical distance away from 0, and bounded above by *m/2*"[63]. Putz further examines this in the following:

bounded above by $m/2$. For the moment, let us suppose that $m/4 \le a \le m/2$. This interval satisfies the conditions at least and has the appeal of simplicity. If we assume that a is randomly distributed, then an estimate of the expected value of a/b is

$$\begin{aligned} E(a/b) &\approx \frac{1}{m/4}\int_{m/4}^{m/2} \frac{x}{m-x}\,dx \\ &= \frac{4}{m}(x + m\ln|x-m|)\Big]_{m/2}^{m/4} \\ &= 4\ln\frac{3}{2} - 1 \\ &\approx 0.6219. \end{aligned}$$

This estimate differs from φ by about 0.6%. Of course, infinitely many other intervals also conform to the assumptions, and the expected values vary widely. For example, $0.3m \le a \le 0.4m$ gives $E(a/b) \approx 0.5415$. The data in Table 1 satisfy $0.348m \le a \le 0.455m$. Using this interval, $E(a/b) \approx 0.6753$. On the other hand, intervals satisfying the conditions can be chosen so that the expected value is exactly φ. One such interval is $[rm, (r+1/5)m]$ where

$$r = \frac{1-(4/5)e^{(\varphi+1)/5}}{1-e^{(\varphi+1)/5}}.$$

Proof of a Reasonable Value for the ratio of a/b[64]

Basically, this shows that even the sonata form has restrictions on this, and these restrictions can cause the ratio of *a/b* to have a tendency to go rather close to 0.618, or rather far from 0.618. That being said, the correlation of the ratio of *a/b*, as seen in the regression line, is still within the range of significance.

Based upon the data drawn, I further my belief that the Golden Ratio was indeed intentional in Mozart's piano sonatas. Even without the data providing a strong pull toward the use of the ratio, Mozart's mathematical background, include his conscious incorporation of mathematical concepts into his music, provides enough insight to suggest that his knowledge of the Golden Ratio was not only possible, but probable. Though some researchers believe it is a coincidence, there is too much evidence in regards to Mozart's relationship with mathematics and number series, let alone the alarming correlation in the data, to believe this was incorporated by coincidence.

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[63] Putz, "The Golden Section and the Piano Sonatas of Mozart," 280.
[64] Putz, "The Golden Section and the Piano Sonatas of Mozart," 280.

Bibliography

Barua, Rajen. “Music, Mathematics and Mozart.” *Gonitsora* (2011).

Einstein, Alfred. *Mozart: His Character, His Work*. New York: Oxford University Press, 1945.

Hunt, Patrick. “Mozart and mathematics.” *Electrum Magazine* (2013).

Irving, John. *Mozart’s Piano Concertos*. Vermont: Ashgate Publishing Limited, 1959.

Irving, John. *Mozart’s Piano Sonatas: Contexts, Sources, Style*. New York: Cambridge University Press, 1997.

Moore, Ray. “Physicist Albert Einstein, seen here playing the violin.”12/8/2014.website. http://ualrpublicradio.org/post/albert-einstein-physicist-and-violinist.

May, Mike. “Did Mozart Use The Golden Section?” *American Scientist* 84 (1996), 118-119.

Niemetschek, Franz: *Life of Mozart*. London: Leonard Hyman, 1798.

Putz, John F. “The Golden Section and the Piano Sonatas of Mozart.” *Mathematics Magazine* 68 (1995), 275-282.

Sollers, Philippe. *Mysterious Mozart*. Illinois: University of Illinois Press, 2010.